Transition Maths and Algebra with Geometry

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Lecture Notes
Electrical and Computer Engineering

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Contents

Well known fields: $\mathbb{Q}$ and $\mathbb{R}$

1. Any two elements a, b can be added. Moreover, $a + b = b + a$ and $(a + b) + c = a + (b + c)$,
2. there is 0, which satisfies for any a, $a + 0 = 0 + a = a$,
3. for any a there is its additive inverse $-a$ such that $a + (-a) = (-a) + a = 0$,
4. any two elements a, b can be multiplied. Moreover, $a \cdot b = b \cdot a$, and $(a \cdot b) \cdot c = a \cdot (b \cdot c)$,
5. there is 1, which satisfies for any a, $a \cdot 1 = 1 \cdot a = a$,
6. for any non-zero element a there is its multiplicative inverse $\frac{1}{a}$, such that $a \cdot \frac{1}{a} = \frac{1}{a} \cdot a = 1$,
7. $(a + b) \cdot c = a \cdot c + b \cdot c$.

Definitions

Definition

A set K together with two binary operations $+$ and $\cdot$ defined on K is called a *field* if it satisfies properties listed below.

- For any $a, b, c \in K$, $a + b = b + a$ and $a + (b + c) = (a + b) + c$,
- there is an element $0 \in K$ such that for any $a \in K$, $a + 0 = 0 + a = a$,
- for any $a \in K$ there exists $-a \in K$ such that $a + (-a) = (-a) + a = 0$,

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Definitions

- for any $a, b, c \in K$, $a \cdot b = b \cdot a$ and $a \cdot (b \cdot c) = (a \cdot b) \cdot c$,
- there is an element $1 \in K$ such that for any $a \in K$, $a \cdot 1 = 1 \cdot a = a$,
- for any $a \in K$ for which $a \neq 0$ there is $\frac{1}{a} \in K$ such that $a \cdot \frac{1}{a} = \frac{1}{a} \cdot a = 1$,
- For any $a, b, c \in K$, $(a + b) \cdot c = a \cdot c + b \cdot c$.

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Non-standard examples

Example: Consider the set $\mathbb{Z}_2 = \{0, 1\}$ and define addition and multiplication by

$$\begin{array}{c|cc} + & 0 & 1 \\ \hline 0 & 0 & 1 \\ 1 & 1 & 0 \end{array} \qquad \begin{array}{c|cc} \cdot & 0 & 1 \\ \hline 0 & 0 & 0 \\ 1 & 0 & 1 \end{array}$$

$\mathbb{Z}_2$ with $+$ and $\cdot$ defined above is a field.

Warning

Not everything with addition and multiplication is a field.

Example: $\mathbb{Z}$ together with the standard addition and multiplication is not a field because only for $1 \in \mathbb{Z}$ there is a multiplicative inverse.

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Complex numbers, informally

Recall that the equation

$$x^2 + 1 = 0$$

doesn't have any real solutions. Sometimes, we need solutions to such equations. What should we do?

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Complex numbers, informally

We should extend the set of reals. Informally, we should introduce a new number

$$i$$

such that $i^2 = -1$. Clearly, i will not be a real number. Introducing i is not enough...

Complex numbers: slightly more formally

Definition

A complex number is an expression $a + bi$, where $a, b \in \mathbb{R}$. The set of all complex numbers is denoted by $\mathbb{C}$.

Definition

We define addition and multiplication by:

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$
$$(a + bi) \cdot (c + di) = (ac - bd) + (ad + bc)i.$$

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Addition and multiplication of complex numbers

Fact

Addition and multiplication of complex numbers are both commutative and associative. Moreover, multiplication is distributive over addition.

Proof: straightforward verification.

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Inversion

For any complex number $a + bi$ we have

$$(a+bi)(a-bi) = a^2 + b^2.$$

Fact

Let $z = a + bi$ be a non-zero complex number. Then the inverse of z is given by

$$\frac{1}{z} = \frac{1}{a+bi} = \frac{a}{a^2+b^2} - \frac{b}{a^2+b^2}i$$

Proof:

$$\frac{1}{a+bi} = \frac{a-bi}{(a+bi)(a-bi)} = \frac{a-bi}{a^2+b^2}$$

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Inversion: examples

The inverse of $z = i$:

$$\frac{1}{i} = \frac{-i}{i \cdot (-i)} = \frac{-i}{1} = -i.$$

The inverse of $z = 1 + i$:

$$\frac{1}{1+i} = \frac{1-i}{(1+i)(1-i)} = \frac{1-i}{2} = \frac{1}{2} + \frac{1}{2}i.$$

Field of complex numbers

Theorem

The set of complex numbers with the complex numbers addition and multiplication forms a field.

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Basic definitions

Definition

Let $z = a + bi$ be a complex number. We define

$$\begin{aligned} \mathfrak{Re}(z) &= a \text{ (the real part of } z\text{)}, \\ \mathfrak{Im}(z) &= b \text{ (the imaginary part of } z\text{)}, \\ |z| &= \sqrt{a^2 + b^2} \text{ (the modulus of } z\text{)}, \\ \bar{z} &= a - bi \text{ (the conjugate of } z\text{)}. \end{aligned}$$

Properties

Properties

- $|z - z_1|$ is the distance between two numbers z and z_1,
- $z \cdot \bar{z} = |z|^2$,
- $|z| = \sqrt{z \cdot \bar{z}}$,
- $\mathfrak{Re}(z) = \frac{1}{2}(z + \bar{z})$
- $\mathfrak{Im}(z) = \frac{1}{2i}(z - \bar{z})$

Proof: an exercise.

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Coordinates

Since any complex number $z = a + bi$ is a pair of two real numbers we can mark its position on the plane.

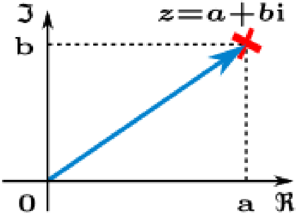

Image source: wikipedia.org

Polar coordinates

Definition

Instead of Cartesian coordinates, we can use polar coordinates to determine a complex number $z = a + bi$. The complex number z is determined by two values: its *modulus* $|z| = \sqrt{a^2 + b^2}$ and its *argument* or *phase* $arg(z)$.

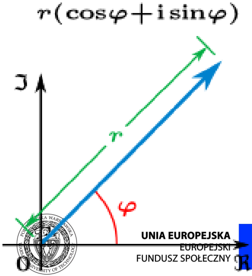

Image source: wikipedia.org

Polar vs. cartesian coordinates

Theorem

Let $z = a + bi$ be a complex number and let $|z|$ be its modulus and $\varphi = \arg(z)$ its argument. Then

$$a = |z|\cos(\varphi),$$
$$b = |z|\sin(\varphi).$$

and

$$|z| = \sqrt{a^2 + b^2},$$

$$\varphi \text{ may be computed from } \tan\varphi = \frac{b}{a}.$$

$$z = |z|\left(\cos\varphi + i\sin\varphi\right)$$

Polar coordinates: properties and examples

Fact

If φ is the argument of a complex number z then so is

$$\varphi + 2k\pi \text{ for any } k \in \mathbb{Z}.$$

Definition

The argument φ of a complex number z for which $\varphi \in (-\pi, \pi]$ is called the *principal argument*.

Let $z = 1 + i$. Then $|z| = \sqrt{2}$. The argument can be deduced from the graph. We have $\arg(z) = \frac{\pi}{4} = 45^o$. In other words, $1 + i = \sqrt{2}(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4})$

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Multiplication in polar coordinates

Theorem

Let $z_1 = |z_1|(\cos\varphi_1 + i\sin\varphi_1)$ and $z_2 = |z_2|(\cos\varphi_2 + i\sin\varphi_2)$ be two complex numbers. Then

$$z_1 \cdot z_2 = |z_1| \cdot |z_2|(\cos(\varphi_1 + \varphi_2) + i\sin(\varphi_1 + \varphi_2))$$

$$\frac{z_1}{z_2} = \frac{|z_1|}{|z_2|}(\cos(\varphi_1 - \varphi_2) + i\sin(\varphi_1 - \varphi_2))$$

De Moivre's formula

Let $z^0 = 1$ and let $z^n = z^{n-1} \cdot z$.

Theorem: De Moivre's formula

Let $z = |z|(\cos\varphi + i\sin\varphi)$ be a complex number and let $n \in \mathbb{N}$. Then

$$z^n = |z|^n(\cos(n\varphi) + i\sin(n\varphi))$$

Example: Calculate $(1+i)^{100}$.

$$(1+i)^{100} = (\sqrt{2}(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}))^{100} =$$

$$\sqrt{2}^{100}(\cos 100\frac{\pi}{4} + i\sin 100\frac{\pi}{4}) = 2^{50}(\cos 25\pi + i\sin 25\pi) =$$

$$2^{50}(\cos\pi + i\sin\pi) = -2^{50}.$$

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De Moivre's formula: consequences

Definition

Any number w satisfying $w^n = z$ for a complex number $z \neq 0$ is called the n-th root of z.

Theorem

Let $z = |z|(\cos\varphi + i\sin\varphi)$ be a complex number and let $n \in \mathbb{N}$. Then every n-th root of z is of the form

$$z_k = \sqrt[n]{|z|}(\cos\frac{\varphi + 2k\pi}{n} + i\sin\frac{\varphi + 2k\pi}{n})$$

for $k = 0, 1, \ldots n-1$.

De Moivre's formula: consequences

Fact

Let z be a complex number. All of the n-th roots of z lie on the circle whose centre is the point 0 and whose radius is $\sqrt[n]{|z|}$. They divide the circumference into n equal arcs.

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De Moivre's formula: example

Example: Find the square roots of $2i$.

$$2i = 2(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}).$$

Then:

$$z_0 = \sqrt{2}(\cos\frac{\frac{\pi}{2}}{2} + i\sin\frac{\frac{\pi}{2}}{2}) = \sqrt{2}(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}) = 1 + i,$$

$$z_1 = \sqrt{2}(\cos\frac{\frac{\pi}{2} + 2\pi}{2} + i\sin\frac{\frac{\pi}{2} + 2\pi}{2}) =$$

$$\sqrt{2}(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}) = -1 - i.$$

Roots of 1

If $z = 1$ then

$$z_k = \cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n}, \text{ for } k \in \{0, \dots n-1\}$$

are the n-th roots of 1.

Contents

Definition

Definition

Let $\mathbb{K}$ be a field. A polynomial f of degree $n =: \deg(f)$ over the field K is an expression of the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$$

for $a_0, a_1, \ldots, a_n \in K$ and $a_n \neq 0$. The set of all polynomials over the field $\mathbb{K}$ is denoted by $\mathbb{K}[x]$. We assume that the degree of the zero polynomial 0 is $-\infty$.

Examples:

$$x + 1, \quad x^2 + (2 + i)x + i, \quad 2x^3 + 1.$$

We define addition and multiplication of two polynomials $f, g \in \mathbb{K}[x]$.

Division

Fact

Let f and g be polynomials over the field $\mathbb{K}$ and let $deg(f) \geq deg(g)$. Then there are polynomials $q, r \in \mathbb{K}[x]$ such that

$$f = q \cdot g + r, \text{ where } deg(r) < deg(g).$$

Example: $f(x) = x^2 + 1, g(x) = x + 1$. Then

$$x^2 + 1 = (x - 1) \cdot (x + 1) + 2$$

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Reducibility

Definition

A polynomial $f \in \mathbb{K}[x]$ is called *reducible* if there are two polynomials $g, h \in \mathbb{K}[x]$ such that $\deg(g) > 0$, $\deg(h) > 0$ and

$$f = g \cdot h.$$

$f \in \mathbb{K}[x]$ is called *irreducible* if for any $g, h \in \mathbb{K}[x]$

$$f = g \cdot h \implies \deg(g) = 0 \text{ or } \deg(h) = 0.$$

Example: $x + 1$ is irreducible over $\mathbb{R}$ and $\mathbb{C}$. $x^2 + 1$ is irreducible over $\mathbb{R}$ but it is reducible over $\mathbb{C}$ since

$$x^2 + 1 = (x - i)(x + i).$$

Main Theorem of Algebra

An element $a \in \mathbb{K}$ is called the root of a polynomial $f \in \mathbb{K}[x]$ if $f(a) = 0$.

Main Theorem of Algebra

Any polynomial $f \in \mathbb{C}[x]$ of degree $deg(f) \geq 1$ has a root in $\mathbb{C}$.

Corollary

Any polynomial $f \in \mathbb{C}[x]$ of degree $deg(f) \geq 1$ can be factored into the product of polynomials of degree 1:

$$f(x) = a(x - z_0)^{k_0}(x - z_1)^{k_1} \dots (x - z_m)^{k_m},$$

where $k_i > 0$ and $k_1 + \dots + k_m = deg(f)$.

Polynomials over complex numbers

Theorem

The formulas for calculating roots of a quadratic polynomial over $\mathbb{C}$ are the same as the formulas used to find roots of quadratic polynomials over $\mathbb{R}$, i.e. given a polynomial $f(x) = ax^2 + bx + c$ where $a, b, c \in \mathbb{C}$ the roots are given by:

$$z_0 = \frac{-b-\delta}{2a},$$
$$z_1 = \frac{-b+\delta}{2a},$$

where δ denotes any square root of $\Delta = b^2 - 4ac$.

Polynomials over reals

Theorem

Let $f \in \mathbb{R}[x]$. If a complex number $z \in \mathbb{C}$ is a root of f then so is its conjugate. In other words,

$$f(z) = 0 \implies f(\bar{z}) = 0.$$

In $\mathbb{C}[x]$ the only irreducible polynomials of positive degree are the polynomials of degree 1.

Corollary

In $\mathbb{R}[x]$ the irreducible polynomials of positive degree are either the polynomials of degree 1 or the polynomials of degree 2 whose $\Delta < 0$.